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An efficient direct solver for multidimensional elliptic Robin boundary value problems using a Legendre spectral-Galerkin method

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ABSTRACT

In this paper, a Legendre–Galerkin method for solving second-order elliptic differential equations subject to the most general nonhomogeneous Robin boundary conditions is presented. The homogeneous Robin boundary conditions are satisfied exactly by expanding the unknown variable using a polynomial basis of functions which are built upon the Legendre polynomials. The direct solution algorithm here developed for the homogeneous Robin problem in two-dimensions relies upon a tensor product process. Nonhomogeneous Robin data are taken into account by means of a lifting. Such a lifting is performed in two successive steps, the first one to account for the data specified at the corners and the second one to account for the boundary values prescribed in the interior of the sides. Numerical results indicating the high accuracy and effectiveness of these algorithms are presented.

1. Introduction

Spectral methods are a widely used tool in the solution of differential equations [1], function approximation and variational problems [2–5]. They involve representing the solution to a problem in terms of truncated series of smooth global functions. They give very accurate approximations for a smooth solution with relatively few degrees of freedom. This accuracy comes about because the spectral coefficients, f_n, typically tend to zero faster than any algebraic power of their index n, showing either exponential or sometimes super-exponential convergence [6]. On the non periodic canonical interval $[-1, 1]$, the Jacobi polynomials are a well-known class of polynomials exhibiting spectral convergence [7–10], of which particular examples are Chebyshev polynomials of the first kind, Chebyshev polynomials of the second kind [11], and Legendre polynomials [12–14]. Chebyshev polynomials of the first kind are equal-ripple (uniform oscillations) and those of the second kind are equal-area (the area under the curve between any two consecutive zeros is constant). Lastly, Legendre polynomials minimize the error between any function and its approximation in the L^2-norm.

Finding a fast and accurate solution of elliptic equations is often an important step in the process of solving problems of fluid dynamics and in other scientific computing applications [15,3,4]. Elliptic equations almost always involve known boundary conditions which can be fully exploited in a Galerkin method [16,1,17,18]. Such a scheme adopts an expansion in terms of a global basis set constructed so that each member explicitly satisfies the boundary conditions. By encoding this additional information, out of all numerical methods, this approach almost always provides the most suitable numerical representation. If an analytic solution of a differential equation is known but difficult to compute, it is expedient to write it in terms of a spectral expansion (for instance in Legendre polynomials) which, once the coefficients are known, is easy to evaluate. In this paper, we shall see such an approximation method can be used to extend the results of [7,19] to elliptic equations with nonhomogeneous Robin boundary conditions in two dimensions.

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Doha and Abd-Elhameed [20] proposed and applied a spectral tau method based on expansion in doubly ultraspherical polynomials for the parabolic and elliptic partial differential equations subject to the most general nonhomogeneous mixed boundary conditions. In [7], the authors presented some efficient Jacobi–Galerkin algorithms for direct solution of second-order differential equations subject to homogeneous Dirichlet boundary conditions based on the matrix decomposition (diagonalization) method [7,21]; while in [19], fast Jacobi–Galerkin algorithms for solving one- and two-dimensional elliptic equations with Neumann boundary conditions are considered.

Auteri et al. [16] introduced an efficient Legendre–Galerkin direct spectral solver for the Neumann problem associated with Laplace and Helmholtz operators in two dimensions that uses a double diagonalization process very similar to that of the Dirichlet spectral solver [22]. The method of [16] is also similar to that of [22] in the way that nonzero boundary values are taken into account. Both methods use a lifting of the boundary data using a two-step procedure, the first step assigning suitable values at the corners and the second assigning the boundary data on the four edges. The point values in the corners will be shown to stem from the derivative of the Neumann datum and are associated with the presence of compatibility conditions between the two slopes in each corner of the domain, as explored by the analysis of Grisvard [23]. Furthermore, Bialecki and Karageorghis [24] proposed a spectral collocation method based on Legendre polynomials for solving the Helmholtz equation in two-dimensions subject to nonhomogeneous Robin boundary conditions. The algorithm in the present work is somewhat related to the ideas used by Auteri and Quartapelle [22], Auteri et al. [16], Doha and Bhrawy [7,8], Doha et al. [19] and Shen [18] in developing fast algorithms for various purposes.

In this paper, we are concerned with the direct solution techniques for second-order elliptic equations subject to nonhomogeneous Robin boundary conditions, using the Legendre–Galerkin approximations (LGM). We present appropriate Legendre basis functions for the approximation of an ordinary differential operator and give the explicit representation of the spectral matrix of the second-order derivative as well as of the mass matrix, including the modes required to impose nonhomogeneous boundary conditions. The direct solution algorithms here developed for the homogeneous Robin problem in two-dimensions rely upon a tensor product process [8,19]. Moreover the treatment of the nonhomogeneous Robin boundary data over a rectangular domain is described, by recalling the concept of lifting the nonzero boundary values. More precisely, such a lifting is performed in two successive steps, the first one to account for the data specified at the corners, in general this step is cumbersome, and the second one to account for the boundary values prescribed in the interior of the sides. The structure of this lifting is similar to that of the two-step procedure proposed in [22] for the Dirichlet boundary value problem and in [16,19] for the Neumann boundary value problem. Numerical results are presented in which the usual exponential convergence behavior of spectral approximations is exhibited.

The content of the paper is organized as follows. In Section 2 we start by introducing the basic concepts. In particular, in Section 2.1 the construction of the basis functions; in Section 2.2 the spectral mass and stiffness matrices are presented and we discuss an algorithm for solving the second-order one-dimensional elliptic equations subject to nonhomogeneous Robin boundary conditions. In Section 3 we describe how problems in two-dimensions with nonhomogeneous Robin boundary conditions can be efficiently transformed into problems with homogeneous Robin boundary conditions. In Section 4, we present various numerical results exhibiting the accuracy and efficiency of our numerical algorithms. We end the paper with a few concluding remarks in Section 5.

2. 1-*D* problem with Robin conditions

In this section, we consider the following one dimensional model problem:

$$\gamma_1\, u - u_{xx} = f(x), \quad \text{in } I = (-1, 1), \tag{2.1}$$

with the Robin type boundary condition

$$\begin{aligned} &a_+ u(1) + b_+ u_x(1) = e_+, \\ &a_- u(-1) + b_- u_x(-1) = e_-, \end{aligned} \tag{2.2}$$

where the given constants $a_+,\ b_+,\ a_-,\ b_-$ are such that

$$(a_+ + 2b_+)a_- - (2a_+ + 3b_+)b_- \neq 0,$$

while $\gamma_1 > 0$ if $a_+ = a_- = 0$ and $\gamma_1 \geq 0$ otherwise, and $f(x)$ is a given source function.

We can split the solution $u(x)$ into the sum of a low degree polynomial which satisfies the nonhomogeneous boundary conditions plus a sum over the basis functions $\phi_j(x)$ that satisfy the equivalent homogeneous boundary conditions.

In such a case we proceed as follows:

Setting

$$u(x) = \widetilde{u}(x) + u_e(x), \tag{2.3}$$

where $\widetilde{u}$ is an auxiliary unknown satisfying a modified equation and with homogeneous Robin boundary conditions at both interval extremes, while $u_e(x)$ is an arbitrary function satisfying the original boundary conditions, $a_\pm u_e(\pm 1) + b_\pm D_x u_e(\pm 1) = e_\pm$.

The modified problem for $\widetilde{u}$ is

$$\begin{aligned}&\gamma_1\widetilde{u}(x) - \widetilde{u}_{xx}(x) = f^*(x), \quad \text{in } I = (-1, 1),\\ &a_\pm\widetilde{u}(\pm 1) + b_\pm\widetilde{u}_x(\pm 1) = 0,\end{aligned} \tag{2.4}$$

where

$$f^*(x) = f(x) - (\gamma_1 - D_{xx})u_e(x). \tag{2.5}$$

Let us first introduce some basic notation which will be used in the sequel. We denote by $L_k(x)$ the kth degree Legendre polynomial, and we set

$$S_N = \text{span}\{L_0(x), L_1(x), \dots, L_{N+2}(x)\},$$
$$W_N = \{v \in S_N : a_\pm v(\pm 1) + b_\pm v_x(\pm 1) = 0\}.$$

Then the standard Legendre–Galerkin approximation to the modified problem (2.4) is: Find $\widetilde{u}_N \in W_N$ such that

$$\gamma_1(\widetilde{u}_N, v) - (D_{xx}\widetilde{u}_N, v) = (f^*, v) \quad \forall\, v \in W_N, \tag{2.6}$$

where $(u, v) = \int_I u(x)\, v(x)\, dx$ is the inner product in $L^2(I)$, whose norm will be denoted by $\|\cdot\|$.

We recall that the $\{L_n(x)\}$ satisfy the orthogonality relation

$$\int_{-1}^{1} L_k(x)L_j(x)\, dx = \begin{cases}0, & k \neq j,\\ h_k, & k = j,\end{cases} \qquad h_k = \frac{2}{2k+1}, \quad \forall\, k, j \geq 0, \tag{2.7}$$

and Rodrigue's formula

$$L_k(x) = \frac{(-1)^k}{2^k k!} D^k\left[(1-x^2)^k\right].$$

We recall also that $L_k(x)$ is a polynomial of degree k and therefore $L_k^{(q)}(x) \in S_{k-q}$. The following relation (the qth derivative of $L_k(x)$) will be needed for our main theorems.

$$L_k^{(q)}(x) = \sum_{\substack{i=0\\ (k+i)\ \text{even}}}^{k-q} C_q(k, i)\, L_i(x), \tag{2.8}$$

where

$$C_q(k, i) = \frac{2^{q-1}(2i+1)\Gamma\left(\frac{1}{2}(q+k-i)\right)\Gamma\left(\frac{1}{2}(q+k+i+1)\right)}{\Gamma(q)\Gamma\left(\frac{1}{2}(2-q+k-i)\right)\Gamma\left(\frac{1}{2}(3-q+k+i)\right)}. \tag{2.9}$$

Some other useful relations are

$$L_k(\pm 1) = (\pm 1)^k, \qquad L_k^{(q)}(\pm 1) = (\pm 1)^{k+q}\frac{k!}{2^q(k-q)!\, q!}\prod_{i=0}^{q-1}(k+i+1). \tag{2.10}$$

2.1. Construction of the basis functions

It is of fundamental importance to note here that the crucial task in applying the Galerkin-spectral approximations is to choose an appropriate basis for W_N such that the linear system resulting from the Legendre–Galerkin approximation to (2.6) is as simple as possible. As suggested in [22,16,19,18], one should choose compact combinations of orthogonal polynomials as basis functions to minimize the bandwidth and condition number of the coefficient matrix corresponding to (2.6).

We start from the formula

$$\phi_k(x, a_\pm, b_\pm) = L_k(x) + \zeta_k(a_\pm, b_\pm)L_{k+1}(x) + \eta_k(a_\pm, b_\pm)L_{k+2}(x), \tag{2.11}$$

where $\zeta_k(a_\pm, b_\pm)$ and $\eta_k(a_\pm, b_\pm)$ are the unique constants such that $\phi_k(x, a_\pm, b_\pm) \in W_N$, $k = 0, 1, \dots, N$. Therefore

$$\zeta_k(a_\pm, b_\pm) = -\frac{2(2k+3)(a_+b_- + b_+a_-)}{2(-2a_- + (k+2)^2 b_-)a_+ + (k+2)^2(-2a_- + (k+1)(k+3)b_-)b_+}, \tag{2.12}$$

$$\eta_k(a_\pm, b_\pm) = -\frac{2(-2a_- + (k+1)^2 b_-)a_+ + (k+1)^2(-2a_- + k(k+2)b_-)b_+}{2(-2a_- + (k+2)^2 b_-)a_+ + (k+2)^2(-2a_- + (k+1)(k+3)b_-)b_+}. \tag{2.13}$$

The first few functions $\phi_k(x, a_\pm, b_\pm)$ of the basis (for various choices of $a_+,\ a_-,\ b_+, b_-$) are drawn in Fig. 1.

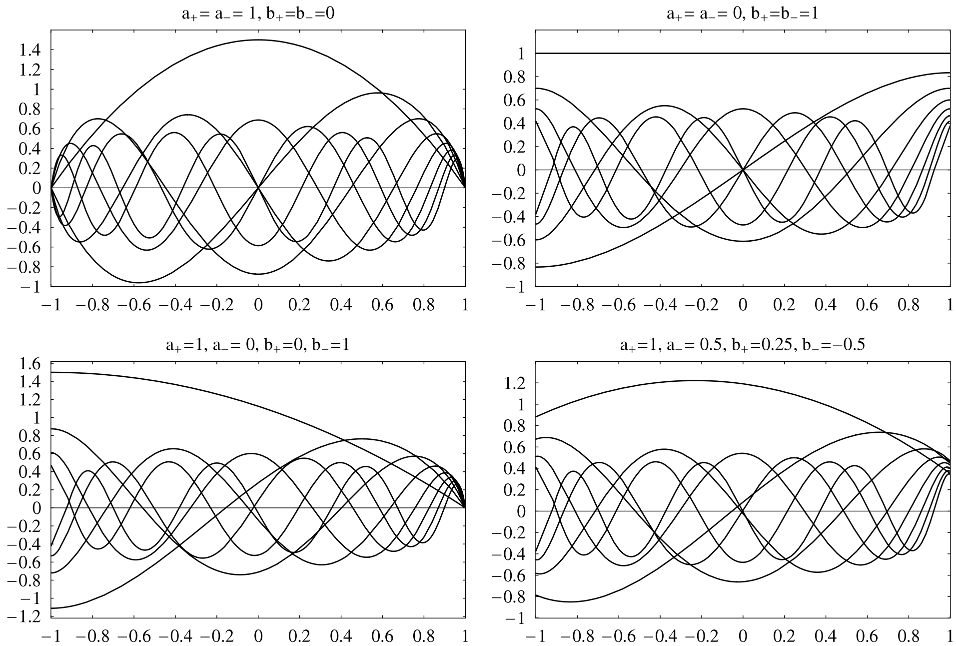

Fig. 1. Functions $\phi_k(x, a_\pm, b_\pm)$ of the basis for one-dimensional boundary value problems for various choices of a_+, a_-, b_+ and b_-.

2.2. Galerkin matrices and solution algorithm

The basis functions $\phi_k(x, a_\pm, b_\pm)$ are chosen such that $\phi_k(x, a_\pm, b_\pm) \in W_N$ for $k = 0, 1, \dots, N$, and it is clear that $\{\phi_k(x, a_\pm, b_\pm)\}_{0\le k\le N}$ are linearly independent and the dimension of W_N is equal to $(N + 1)$. Hence,

$$W_N = \text{span}\{\phi_0(x, a_\pm, b_\pm), \phi_1(x, a_\pm, b_\pm), \dots, \phi_N(x, a_\pm, b_\pm)\}.$$

The stiffness matrix $A^{a\pm}_{b\pm}$ is defined from the bilinear form:

$$-\big(\phi_j''(x, a_\pm, b_\pm), \phi_k(x, a_\pm, b_\pm)\big) = -\int_{-1}^{1} \phi_j''(x, a_\pm, b_\pm)\phi_k(x, a_\pm, b_\pm)\, dx,$$

and its elements are

$$a_{kj}(a_\pm, b_\pm) = -\left(\phi_j''(x, a_\pm, b_\pm), \phi_k(x, a_\pm, b_\pm)\right), \quad k, j \ge 0. \tag{2.14}$$

Immediately, if we set $q = 2$ in relation (2.8), we obtain

$$D^2 L_j(x) = \sum_{\substack{i=0\\(j+i)\ even}}^{j-2} C_2(j, i) L_i(x), \tag{2.15}$$

where

$$C_2(j, i) = \left(i + \frac{1}{2}\right)(j - i)(j + i + 1),$$

and using (2.15), we have

$$D^2\phi_j(x, a_\pm, b_\pm) = \sum_{\substack{i=0\\(j+i)\ \text{even}}}^{j-2} C_2(j, i)L_i(x) + \zeta_j(a_\pm, b_\pm) \sum_{\substack{i=0\\(j+i)\ \text{even}}}^{j-1} C_2(j + 1, i)L_i(x)$$
$$+ \eta_j(a_\pm, b_\pm) \sum_{\substack{i=0\\(j+i)\ \text{even}}}^{j} C_2(j + 2, i)L_i(x). \tag{2.16}$$

By the orthogonality of the Legendre polynomials (2.7) and using (2.16) and (2.11), we immediately observe that $a_{kj}(a_\pm, b_\pm) = 0$, for $k \neq j$, while for $k = j$ a direct calculation gives

$$\begin{aligned} a_{kk}(a_\pm, b_\pm) &= -\eta_k\,(a_\pm, b_\pm)\,C_2(k+2, k)h_k \\ &= -\frac{2(2k+3)\big(2a_-(2a_+ + (k+1)^2 b_+) - (k+1)^2 b_-(2a_+ + k(k+2)b_+)\big)}{-2a_-(2a_+ + (k+2)^2 b_+) + (k+2)^2 b_-(2a_+ + (k+1)(k+3)b_+)}. \end{aligned} \tag{2.17}$$

The mass matrix $B^{a_\pm}_{b_\pm}$ defined by the bilinear form $b_{kj}(a_\pm, b_\pm) = \big(\phi_j(x, a_\pm, b_\pm), \phi_k(x, a_\pm, b_\pm)\big)$ is symmetric and pentadiagonal. It can be shown, with the aid of (2.11) and (2.7), that $b_{kj}(a_\pm, b_\pm)$ are different from zero only for $j = k + r - 2$, $r = 0, 1, 2, 3, 4$ and are given by the following formulae

$$\left.\begin{aligned} &b_{kk}(a_\pm, b_\pm) = h_k + \zeta_k^2(a_\pm, b_\pm)h_{k+1} + \eta_k^2(a_\pm, b_\pm)h_{k+2}, \\ &b_{k,k+1}(a_\pm, b_\pm) = b_{k+1,k}(a_\pm, b_\pm) = \zeta_k(a_\pm, b_\pm)h_{k+1} + \eta_k(a_\pm, b_\pm)\zeta_{k+1}(a_\pm, b_\pm)h_{k+2}, \\ &b_{k,k+2}(a_\pm, b_\pm) = b_{k+2,k}(a_\pm, b_\pm) = \eta_k(a_\pm, b_\pm)h_{k+2}. \end{aligned}\right\} \tag{2.18}$$

The function $u_e(x)$ satisfying the original boundary conditions, $a_\pm u_e(\pm 1) + b_\pm D_x u_e(\pm 1) = e_\pm$, is taken to be the form

$$u_e(x) = L_0(x) + \beta_1 L_1(x) + \beta_2\, L_2(x), \tag{2.19}$$

so that one obtains immediately

$$\beta_1 = \frac{3(a_- - e_-)b_+ - (e_- - 3b_-)a_+ + (a_- - 3b_-)e_+}{2(a_+ + 2b_+)a_- - 2(2a_+ + 3b_+)b_-},$$

$$\beta_2 = -\frac{(a_- - e_-)(a_+ + b_+) - (e_+ - a_+)(a_- - b_-)}{2(a_+ + 2b_+)a_- - 2(2a_+ + 3b_+)b_-}.$$

Now it is not difficult to show, by using (2.5) and (2.19), that

$$f^*(x) = f(x) - (\gamma_1 - 3\beta_2)\, L_0(x) - \gamma_1\beta_1\, L_1(x) - \gamma_1\beta_2\, L_2(x).$$

It is now clear that the variational formulation of (2.6) is equivalent to

$$\gamma_1\Big(\widetilde{u}_N,\ \phi_k(x, a_\pm, b_\pm)\Big) - \Big(D_{xx}\widetilde{u}_N,\ \phi_k(x, a_\pm, b_\pm)\Big) = \Big(f^*(x),\ \phi_k(x, a_\pm, b_\pm)\Big), \quad k = 0, 1, \ldots, N. \tag{2.20}$$

Let us denote

$$f_k = \big(f, \phi_k(x, a_\pm, b_\pm)\big), \qquad f_k^* = (f^*, \phi_k(x, a_\pm, b_\pm)), \qquad \mathbf{f}^* = (f_0^*, f_1^*, \ldots, f_N^*)^T,$$

$$\widetilde{u}_N(x) = \sum_{k=0}^{N} a_k\phi_k(x, a_\pm, b_\pm), \qquad \mathbf{a} = \big(a_0, a_1, \ldots, a_N\big)^T, \qquad A^{a_\pm}_{b_\pm} = (a_{kj}(a_\pm, b_\pm)),$$

$$B^{a_\pm}_{b_\pm} = (b_{kj}(a_\pm, b_\pm)), \quad 0 \le k, j \le N.$$

Then (2.20) is equivalent to the following matrix equation

$$(A^{a_\pm}_{b_\pm} + \gamma_1 B^{a_\pm}_{b_\pm})\,\mathbf{a} = \mathbf{f}^*, \tag{2.21}$$

where $A^{a_\pm}_{b_\pm}$ and $B^{a_\pm}_{b_\pm}$ are the stiffness and mass matrices, while f_k^* is given explicitly by

$$f_k^* = \begin{cases} f_0 - (\gamma_1 - 3\beta_2)h_0 - \gamma_1\beta_1\zeta_0(a_\pm, b_\pm)h_1 - \gamma_1\beta_2\eta_0(a_\pm, b_\pm)h_2, & k = 0, \\ f_1 - \gamma_1\beta_1\, h_1 - \gamma_1\beta_2\zeta_1(a_\pm, b_\pm)\, h_2, & k = 1, \\ f_2 - \gamma_1\beta_2\, h_2, & k = 2, \\ f_k, & k = 3, 4, \ldots, N. \end{cases}$$

Finally, the complete solution is written in the form

$$u_N(x) = \sum_{k=0}^{N} a_k\phi_k(x, a_\pm, b_\pm) + L_0(x) + \beta_1 L_1(x) + \beta_2\, L_2(x).$$

In the case of Dirichlet boundary conditions ($a_\pm \neq 0,\ b_\pm = 0$) or Neumann boundary conditions ($a_\pm = 0,\ b_\pm \neq 0$), and since $b_{kj}(a_\pm, b_\pm) = 0$ for $k \neq j$ and $k \neq j - 2$, then we observe that $B^{a_\pm}_{b_\pm}$ (respectively the system (2.21) with $\gamma_1 \neq 0$) can be decoupled into two tridiagonal submatrices (respectively two tridiagonal subsystems for the odd and even components of **a**). Moreover, and in the general case of Robin boundary conditions ($a_\pm \neq 0,\ b_\pm \neq 0$), the linear system (2.21) with $\gamma_1 \neq 0$, can be solved by forming explicitly the LUfactorization; i.e., $A^{a_\pm}_{b_\pm} + \gamma_1 B^{a_\pm}_{b_\pm} = LU$. The special structure of L and U enables us to obtain the solution in $O(N)$ operations. We notice also that the system (2.21) reduces to a diagonal system for $\gamma_1 = 0$.

It is noted that the result (that $a_{kj}(a_\pm, b_\pm)$ is diagonal) does not extend to other types of orthogonal polynomial: e.g. if the basis functions are built from Chebyshev rather than Legendre polynomials, this result (with the appropriate weight) fails to hold, the reason why this holds remains unknown (see, [1] for more details).

3. 2-*D* elliptic problem with Robin boundary conditions

Let us now consider the two-dimensional boundary value problem for the Helmholtz operator supplemented by Robin conditions, namely

$$\gamma_1 u(x,y) - \Delta u(x,y) = f(x,y) \quad \text{in } \Omega, \tag{3.1}$$

and

$$\begin{aligned} &a_+\, u(1,y) + b_+\, u_x(1,y) = e^r(y), \qquad a_-\, u(-1,y) + b_-\, u_x(-1,y) = e^\ell(y), \\ &c_+\, u(x,1) + d_+\, u_y(x,1) = e^t(x), \qquad c_-\, u(x,-1) + d_- u_y(x,-1) = e^b(x), \end{aligned} \tag{3.2}$$

where the given constants $a_+,\ b_+,\ a_-,\ b_-,\ c_+,\ d_+,\ c_-,\ d_-$ are such that

$$\begin{aligned} &c_-(2c_+ + 3d_+) - d_-(3c_+ + 4d_+) \neq 0, \\ &a_-(2a_+ + 3b_+) - b_-(3a_+ + 4b_+) \neq 0, \end{aligned}$$

while $e^r(y),\ e^\ell(y),\ e^t(x)$ and $e^b(x)$ are known functions in their arguments, $\Omega = I \times I$, the differential operator Δ is the well-known Laplacian defined by $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and $f(x,y)$ is a given source function. Here, $\gamma_1 > 0$ if $a_\pm = c_\pm = 0$ and $\gamma_1 \geq 0$ otherwise.

In the following we describe how problems with nonhomogeneous boundary conditions can be efficiently transformed into problems with homogeneous boundary conditions.

We proceed as follows:

Setting

$$u(x,y) = \widetilde{u}(x,y) + u_e(x,y),$$

where $\widetilde{u}$ is an auxiliary unknown function satisfying the modified problem

$$\gamma_1 \widetilde{u}(x,y) - \Delta \widetilde{u}(x,y) = f^*(x,y) \quad \text{in } \Omega, \tag{3.3}$$

subject to the homogeneous boundary conditions

$$\begin{aligned} &a_\pm\, \widetilde{u}(\pm 1, y) + b_\pm\, \widetilde{u}_x(\pm 1, y) = 0, \\ &c_\pm\, \widetilde{u}(x, \pm 1) + d_\pm\, \widetilde{u}_y(x, \pm 1) = 0, \end{aligned} \tag{3.4}$$

where $f^*(x,y) = f(x,y) - (\gamma_1 - \Delta)u_e(x,y)$, while $u_e(x,y)$ is an arbitrary function satisfying the original boundary conditions (3.2).

3.1. Legendre–Galerkin approximation

We introduce the space $V_N = W_N \otimes Z_N$ where

$$\begin{aligned} &W_N = \{\nu \in S_N : a_\pm \nu(\pm 1) + b_\pm \nu_x(\pm 1) = 0\}, \\ &Z_N = \{\nu \in S_N : c_\pm \nu(\pm 1) + d_\pm \nu_y(\pm 1) = 0\}. \end{aligned}$$

Let $\{\phi_k(x, a_\pm, b_\pm)\}_{k=0}^N,\ x \in I$ be the basis for W_N such that

$$\phi_k(x, c_\pm, d_\pm) = L_k(x) + \zeta_k(a_\pm, b_\pm) L_{k+1}(x) + \eta_k(a_\pm, b_\pm) L_{k+2}(x)$$

where $\zeta_k(a_\pm, b_\pm)$ and $\eta_k(a_\pm, b_\pm)$ are as defined by (2.12) and (2.13).

Let $\{\phi_j(y, c_\pm, d_\pm)\}_{j=0}^N,\ y \in I$ be the basis for Z_N such that

$$\phi_j(y, c_\pm, d_\pm) = L_j(y) + \zeta_j(c_\pm, d_\pm) L_{j+1}(y) + \eta_j(c_\pm, d_\pm) L_{j+2}(y)$$

where $\zeta_k(c_\pm, d_\pm)$ and $\eta_k(c_\pm, d_\pm)$ are defined by

$$\zeta_j(c_\pm, d_\pm) = -\frac{2(2j+3)(c_+ d_- + d_+ c_-)}{2(-2c_- + (j+2)^2 d_-)c_+ + (j+2)^2(-2c_- + (j+1)(j+3)d_-)d_+}, \tag{3.5}$$

and

$$\eta_j(c_\pm, d_\pm) = -\frac{2(-2c_- + (j+1)^2 d_-)c_+ + (j+1)^2(-2c_- + j(j+2)d_-)d_+}{2(-2c_- + (j+2)^2 d_-)c_+ + (j+2)^2(-2c_- + (j+1)(j+3)d_-)d_+}. \tag{3.6}$$

Then it is obvious that

$$V_N = \text{span}\{\phi_i(x, a_\pm, b_\pm)\, \phi_j(y, c_\pm, d_\pm),\ i,j = 0, 1, \ldots, N\} \tag{3.7}$$

and the standard Legendre–Galerkin approximation of (3.3)–(3.4) consists of finding $\widetilde{u}_N \in V_N$ such that

$$\gamma_1\big(\widetilde{u}_N, v\big) - \big(\Delta\widetilde{u}_N, v\big) = \big(f^*, v\big), \quad \forall v \in V_N. \tag{3.8}$$

The approximate solution u_N to u is then expressed as the sum of two components:

$$u_N(x, y) = \widetilde{u}_N(x, y) + u_{e,N}(x, y) = \sum_{k=0}^{N}\sum_{j=0}^{N} u_{kj}\,\phi_k(x, a_\pm, b_\pm)\,\phi_j(y, c_\pm, d_\pm) + u_{e,N}(x, y). \tag{3.9}$$

The precise form of the expansion of $u_{e,N}(x, y)$ in terms of the polynomial basis will be given in Section 3.3.

3.2. Compatibility conditions of the Robin boundary values

The distribution of the Robin datum on the bottom, top, left and right sides are not completely independent since they must satisfy the following four compatibility conditions in the corners:

$$\left.\begin{aligned}
&c_+e^r(1) + d_+D_ye^r(1) = a_+e^t(1) + b_+D_xe^t(1),\\
&c_-e^r(-1) + d_-D_ye^r(-1) = a_+e^b(1) + b_+D_xe^b(1),\\
&c_+e^\ell(1) + d_+D_ye^\ell(1) = a_-e^t(-1) + b_-D_xe^t(-1),\\
&c_-e^\ell(-1) + d_-D_ye^\ell(-1) = a_-e^b(-1) + b_-D_xe^b(-1).
\end{aligned}\right\} \tag{3.10}$$

These four relations correspond to the conditions of equality of the following four conditions of the unknown u in the corners:

$$\left.\begin{aligned}
&(a_+ + b_+\,D_x)(c_+ + d_+\,D_y)\,u(1, 1) = (c_+ + d_+\,D_y)(a_+ + b_+\,D_x)\,u(1, 1),\\
&(a_+ + b_+D_x)(c_- + d_-\,D_y)\,u(1, -1) = (c_- + d_-\,D_y)(a_+ + b_+D_x)\,u(1, -1),\\
&(a_- + b_-\,D_x)(c_+ + d_+\,D_y)\,u(-1, 1) = (c_+ + d_+\,D_y)(a_- + b_-\,D_x)\,u(-1, 1),\\
&(a_- + b_-\,D_x)(c_- + d_-\,D_y)\,u(-1, -1) = (c_- + d_-\,D_y)(a_- + b_-\,D_x)\,u(-1, -1).
\end{aligned}\right\} \tag{3.11}$$

For the development of the solution algorithm, it is convenient to denote explicitly these four corner values as follows:

$$\left.\begin{aligned}
&c^{r\,t} \equiv c_+e^r(1) + d_+D_ye^r(1) = a_+e^t(1) + b_+D_xe^t(1),\\
&c^{r\,b} \equiv c_-e^r(-1) + d_-D_ye^r(-1) = a_+e^b(1) + b_+D_xe^b(1),\\
&c^{\ell\,t} \equiv c_+e^\ell(1) + d_+D_ye^\ell(1) = a_-e^t(-1) + b_-D_xe^t(-1),\\
&c^{\ell\,b} \equiv c_-e^\ell(-1) + d_-D_ye^\ell(-1) = a_-e^b(-1) + b_-D_xe^b(-1).
\end{aligned}\right\} \tag{3.12}$$

3.3. Lifting of nonhomogeneous boundary values

Now the function $u_{e,N}(x, y)$ approximating the analytical lifting $u_e(x, y)$ is decomposed in two contributions as follows:

$$u_{e,N}(x, y) = u_e^c(x, y) + u_{e,N}^s(x, y). \tag{3.13}$$

Here, $u_e^c(x, y)$ is the corner component dependent on the four values $c^{lb},\ c^{rb},\ c^{lt}$ and c^{rt} defined by (3.12), while $u_e^s(x, y)$ is the side component that accounts for the values of the Robin datum inside each of the sides of the domain.

3.3.1. Corner component of the lifting

The corner component $u_e^c(x, y)$ of the lifting is expressed by means of the polynomial

$$u_e^c(x, y) = L_0(x)L_0(y) + \alpha_1\,L_1(x)L_1(y) + \alpha_2\,L_2(x)L_1(y) + \alpha_3\,L_1(x)L_2(y) + \alpha_4\,L_2(x)L_2(y), \tag{3.14}$$

where the coefficients $\alpha_1,\ \alpha_2,\ \alpha_3$ and α_4 are determined by exploiting the four relations (3.12) and enforcing the following conditions, in a pointwise manner

$$\left.\begin{aligned}
&(a_+ + b_+\,D_x)(c_+ + d_+\,D_y)\,u_e^c(1, 1) = c^{r\,t},\\
&(a_+ + b_+D_x)(c_- + d_-\,D_y)\,u_e^c(1, -1) = c^{r\,b},\\
&(a_- + b_-\,D_x)(c_+ + d_+\,D_y)\,u_e^c(-1, 1) = c^{\ell\,t},\\
&(a_- + b_-\,D_x)(c_- + d_-\,D_y)\,u_e^c(-1, -1) = c^{\ell\,b}.
\end{aligned}\right\} \tag{3.15}$$

Hence, the coefficients $\{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4\}$ can be determined from (3.15).

3.3.2. Side component of the lifting

Once the corner component of the lifting u_e^c has been evaluated, the side component $u_{e,N}^s$ is determined so as to satisfy (in a weak sense, see later) Robin boundary conditions with respect to the perturbed datum

$$\begin{aligned}\widetilde{e}^{\binom{t}{b}}(x) &= c_\pm\, u_e^s(x,\pm 1) + d_\pm\, D_y\, u_e^s(x,\pm 1)\\ &\equiv e^{\binom{t}{b}}(x) - c_\pm\, u_e^c(x,\pm 1) - d_\pm\, D_y\, u_e^c(x,\pm 1),\end{aligned} \tag{3.16}$$

$$\begin{aligned}\widetilde{e}^{\binom{r}{\ell}}(y) &= a_\pm\, u_e^s(\pm 1, y) + b_\pm\, D_x\, u_e^s(\pm 1, y)\\ &\equiv e^{\binom{r}{\ell}}(y) - a_\pm\, u_e^c(\pm 1, y) - b_\pm\, D_x\, u_e^c(\pm 1, y),\end{aligned} \tag{3.17}$$

with the superscripts within parentheses to be selected alternatively according to the signs $\pm$ written on the right. By (3.14), the distribution of these perturbed Robin conditions on the four sides is given explicitly by

$$\widetilde{e}^{\binom{t}{b}}(x) = e^{\binom{t}{b}}(x) - \epsilon_0^{\binom{t}{b}} L_0(x) - \epsilon_1^{\binom{t}{b}} L_1(x) - \epsilon_2^{\binom{t}{b}} L_2(x), \tag{3.18}$$

$$\widetilde{e}^{\binom{r}{\ell}}(y) = e^{\binom{r}{\ell}}(y) - \epsilon_0^{\binom{r}{\ell}} L_0(y) - \epsilon_1^{\binom{r}{\ell}} L_1(y) - \epsilon_2^{\binom{r}{\ell}} L_2(y), \tag{3.19}$$

where

$$\epsilon_j^t = \begin{cases} c_+, & j=0,\\ (c_+ + d_+)\alpha_1 + (c_+ + 3d_+)\alpha_3, & j=1,\\ (c_+ + d_+)\alpha_2 + (c_+ + 3d_+)\alpha_4, & j=2,\end{cases}$$

$$\epsilon_j^b = \begin{cases} c_-, & j=0,\\ (-c_- + d_-)\alpha_1 + (c_- - 3d_-)\alpha_3, & j=1,\\ (-c_- + d_-)\alpha_2 + (c_- - 3d_-)\alpha_4, & j=2,\end{cases}$$

$$\epsilon_j^r = \begin{cases} a_+, & j=0,\\ (a_+ + b_+)\alpha_1 + (a_+ + 3b_+)\alpha_2, & j=1,\\ (a_+ + b_+)\alpha_3 + (a_+ + 3b_+)\alpha_4, & j=2,\end{cases}$$

$$\epsilon_j^\ell = \begin{cases} a_-, & j=0,\\ (-a_- + b_-)\alpha_1 + (a_- - 3b_-)\alpha_2, & j=1,\\ (-a_- + b_-)\alpha_3 + (a_- - 3b_-)\alpha_4, & j=2.\end{cases}$$

Throughout this paper we use the symbol $\otimes$ to denote both the tensor product of matrices and the tensor product of function spaces. To represent the (approximated) side component $u_{e,N}^s(x,y)$ of the lifting we introduce the space

$$\left[\{\phi_k(x, a_\pm, b_\pm), 0 \le k \le N\} \otimes \{y, y^2\}\right] \oplus \left[\{\phi_j(y, c_\pm, d_\pm), 0 \le j \le N\} \otimes \{x, x^2\}\right]$$

where $\otimes$ and $\oplus$ are the outer product and direct sum of basis functions, so that we have the following expansion:

$$u_{e,N}^s(x,y) = \left[\sum_{k=0}^{N} \left(d_k^{\mathrm{I}}\, y + d_k^{\mathrm{II}}\, y^2\right)\phi_k(x, a_\pm, b_\pm)\right] + \left[\sum_{j=0}^{N} \left(d_j^{\mathrm{III}}\, x + d_j^{\mathrm{IV}}\, x^2\right)\phi_j(y, c_\pm, d_\pm)\right], \tag{3.20}$$

with the following conditions at horizontal and vertical sides,

$$\begin{aligned} c_\pm\, u_{e,N}^s(x,\pm 1) + d_\pm\, D_y\, u_{e,N}^s(x,\pm 1) &= \sum_{k=0}^{N}\left(d_k^{\mathrm{I}}(\pm c_\pm + d_\pm) + d_k^{\mathrm{II}}(c_\pm \pm 2d_\pm)\right)\phi_k(x, a_\pm, b_\pm)\\ &\quad + \sum_{j=0}^{N}\left(x\, d_j^{\mathrm{III}} + x^2\, d_j^{\mathrm{IV}}\right)\left(c_\pm\, \phi_j(\pm 1, c_\pm, d_\pm) + d_\pm\, D_y\, \phi_j(\pm 1, c_\pm, d_\pm)\right)\\ &= \sum_{k=0}^{N}\left(d_k^{\mathrm{I}}(\pm c_\pm + d_\pm) + d_k^{\mathrm{II}}(c_\pm \pm 2d_\pm)\right)\phi_k(x, a_\pm, b_\pm), \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} a_\pm\, u^s_{e,N}(\pm 1, y) + b_\pm\, D_x\, u^s_{e,N}(\pm 1, y) &= \sum_{j=0}^{N}\Big(d_j^{\mathrm{III}}(\pm\, a_\pm + b_\pm) + \, d_j^{\mathrm{IV}}(a_\pm \pm\, 2b_\pm)\Big)\phi_j(y, c_\pm, d_\pm) \\ &\quad + \sum_{k=0}^{N}\Big(y\, d_k^{\mathrm{I}} + y^2\, d_k^{\mathrm{II}}\Big)\Big(a_\pm\, \phi_k(\pm 1, a_\pm, b_\pm) + b_\pm\, D_x\, \phi_k(\pm 1, a_\pm, b_\pm)\Big) \\ &= \sum_{j=0}^{N}\Big(d_j^{\mathrm{III}}(\pm\, a_\pm + b_\pm) + \, d_j^{\mathrm{IV}}(a_\pm \pm\, 2b_\pm)\Big)\phi_j(y, c_\pm, d_\pm), \end{aligned} \tag{3.22}$$

respectively, where $\{d_k^i;\ \ i = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}$ are the coefficients to be determined. We now consider the horizontal sides and impose that the Robin boundary conditions in the y-direction (3.21) (for $y = \pm 1$) be equal, in the sense of L^2 projection, to $\widetilde{e}^{\,t}(x)$ and $\widetilde{e}^{\,b}(x)$; that is, we require that

$$\begin{aligned} &\Big(c_\pm\, u^s_{e,N}(x, \pm 1) + d_\pm\, D_y\, u^s_{e,N}(x, \pm 1), \phi_i(x, a_\pm, b_\pm)\Big) \\ &\quad = \sum_{k=0}^{N}\Big(d_k^{\mathrm{I}}(\pm\, c_\pm + d_\pm) + \, d_k^{\mathrm{II}}(c_\pm \pm\, 2d_\pm)\Big)\Big(\phi_k(x, a_\pm, b_\pm), \phi_i(x, a_\pm, b_\pm)\Big) \\ &\quad = \Big(\widetilde{e}^{\,\binom{t}{b}}(x), \phi_i(x, a_\pm, b_\pm)\Big). \end{aligned} \tag{3.23}$$

In a similar way, for the vertical sides, (3.19) and (3.22) imply that

$$\begin{aligned} &\Big(a_\pm\, u^s_{e,N}(\pm 1, y) + b_\pm\, D_x\, u^s_{e,N}(\pm 1, y), \phi_i(y, c_\pm, d_\pm)\Big) \\ &\quad = \sum_{j=0}^{N}\Big(d_j^{\mathrm{III}}(\pm\, a_\pm + b_\pm) + \, d_j^{\mathrm{IV}}(a_\pm \pm\, 2b_\pm)\Big)\Big(\phi_j(y, c_\pm, d_\pm), \phi_i(y, c_\pm, d_\pm)\Big) \\ &\quad = \Big(\widetilde{e}^{\,\binom{r}{\ell}}(y), \phi_i(y, c_\pm, d_\pm)\Big). \end{aligned} \tag{3.24}$$

It is now clear that (3.23) and (3.24) are equivalent to the following linear systems:

$$\left.\begin{aligned} B^{a_\pm}_{b_\pm}\Bigg((c_+ + d_+)\mathbf{d}^{\mathrm{I}} + (c_+ + 2d_+)\mathbf{d}^{\mathrm{II}}\Bigg) = \mathbf{s}^{\,t}, \\ B^{a_\pm}_{b_\pm}\Bigg((d_- - c_-)\mathbf{d}^{\mathrm{I}} + (c_- - 2d_-)\mathbf{d}^{\mathrm{II}}\Bigg) = \mathbf{s}^{\,b}, \end{aligned}\right\} \tag{3.25}$$

$$\left.\begin{aligned} B^{c_\pm}_{d_\pm}\Bigg((a_+ + b_+)\mathbf{d}^{\mathrm{III}} + (a_+ + 2b_+)\mathbf{d}^{\mathrm{IV}}\Bigg) = \mathbf{s}^{\,r}, \\ B^{c_\pm}_{d_\pm}\Bigg((b_- - a_-)\mathbf{d}^{\mathrm{III}} + (a_- - 2b_-)\mathbf{d}^{\mathrm{IV}}\Bigg) = \mathbf{s}^{\,\ell}, \end{aligned}\right\} \tag{3.26}$$

where $B^{a_\pm}_{b_\pm},\ B^{c_\pm}_{d_\pm}$ are the mass matrices, $\big\{\mathbf{d}^\sigma = (d_0^\sigma, d_1^\sigma, \ldots, d_N^\sigma)^T\, ;\ \sigma = \mathrm{I},\ \mathrm{II},\ \mathrm{III and\ IV}\big\}$ are vectors of unknown expansion coefficient and $\big\{\mathbf{s}^\sigma = (s_0^\sigma, s_1^\sigma, \ldots, s_N^\sigma)^T\, ; \sigma = t,\ b,\ r \text{ and } l\big\}$ are the column vectors of known source terms. The linear systems for $\{\mathbf{d}^{\mathrm{I}},\ \mathbf{d}^{\mathrm{II}}\}$ and $\{\mathbf{d}^{\mathrm{III}},\ \mathbf{d}^{\mathrm{IV}}\}$ give immediately the following four linear systems:

$$\begin{aligned} B^{a_\pm}_{b_\pm}\, \mathbf{d}^{\mathrm{I}} &= \frac{(c_- - 2d_-)\mathbf{s}^{\,t} - (c_+ + 2d_+)\mathbf{s}^{\,b}}{c_-(2c_+ + 3d_+) - d_-(3c_+ + 4d_+)}, \\ B^{a_\pm}_{b_\pm}\, \mathbf{d}^{\mathrm{II}} &= \frac{(c_- - d_-)\mathbf{s}^{\,t} + (c_+ + d_+)\mathbf{s}^{\,b}}{c_-(2c_+ + 3d_+) - d_-(3c_+ + 4d_+)}, \\ B^{c_\pm}_{d_\pm}\, \mathbf{d}^{\mathrm{III}} &= \frac{(a_- - 2b_-)\mathbf{s}^{\,r} - (a_+ + 2b_+)\mathbf{s}^{\,\ell}}{a_-(2a_+ + 3b_+) - b_-(3a_+ + 4b_+)}, \\ B^{c_\pm}_{d_\pm}\, \mathbf{d}^{\mathrm{IV}} &= \frac{(a_- - b_-)\mathbf{s}^{\,r} + (a_+ + b_+)\mathbf{s}^{\,\ell}}{a_-(2a_+ + 3b_+) - b_-(3a_+ + 4b_+)}. \end{aligned}$$

Now, the expansion of any $e^\sigma \in L^2(-1,1)$ (the original Robin boundary conditions) in terms of $L_n(z)$'s is

$$e^\sigma(z) = \sum_{n=0}^{\infty} m_n^\sigma L_n(z); \quad \sigma = t,\ b,\ r \text{ and } l, \tag{3.27}$$

where

$$m_n^\sigma = \frac{(-1)^n}{2^n n! h_n} \int_{-1}^{1} e^\sigma(z) D^n\big[(1-z^2)^n\big]dz,$$

and after integrating the above equation by parts n times, it follows that

$$m_n^\sigma = \frac{1}{2^n n! h_n} \int_{-1}^{1} D^n\big[e^\sigma(z)\big](1-z^2)^n dz,$$

(cf., [3, Section. 2.3.1]). Since the right-hand sides of (3.23) are defined by $s_i^t = \big(\widetilde{e}^t(x), \phi_i(x, a_\pm, b_\pm)\big)$ and $s_i^b = \big(\widetilde{e}^b(x), \phi_i(x, a_\pm, b_\pm)\big)$, also the right-hand sides of (3.24) are defined by $s_i^r = \big(\widetilde{e}^r(y), \phi_i(y, c_\pm, d_\pm)\big)$ and $s_i^l = \big(\widetilde{e}^\ell(y), \phi_i(y, c_\pm, d_\pm)\big)$. Then by using (3.27), (3.18), (3.19), (2.7) and the definition of the basis functions in (2.11), we immediately observe that the nonzero elements of s_j^σ; $\sigma = t,\ b,\ r$ and l are given explicitly by

$$s_j^{\binom{t}{b}} = \begin{cases} (m_0^{\binom{t}{b}} - \epsilon_0^{\binom{t}{b}})h_0 + (m_1^{\binom{t}{b}} - \epsilon_1^{\binom{t}{b}})\zeta_0(a_\pm, b_\pm)h_1 + (m_2^{\binom{t}{b}} - \epsilon_2^{\binom{t}{b}})\eta_0(a_\pm, b_\pm)h_2, & j = 0, \\ (m_1^{\binom{t}{b}} - \epsilon_1^{\binom{t}{b}})h_1 + (m_2^{\binom{t}{b}} - \epsilon_2^{\binom{t}{b}})\zeta_1(a_\pm, b_\pm)h_2 + m_3^{\binom{t}{b}}\eta_1(a_\pm, b_\pm)h_3, & j = 1, \\ (m_2^{\binom{t}{b}} - \epsilon_2^{\binom{t}{b}})h_2 + m_3^{\binom{t}{b}}\zeta_2(a_\pm, b_\pm)h_3 + m_4^{\binom{t}{b}}\eta_2(a_\pm, b_\pm)h_4, & j = 2, \\ m_j^{\binom{t}{b}}h_j + m_{j+1}^{\binom{t}{b}}\zeta_j(a_\pm, b_\pm)h_{j+1} + m_{j+2}^{\binom{t}{b}}\eta_j(a_\pm, b_\pm)h_{j+2}, & j = 3, \dots, N, \end{cases}$$

$$s_j^{\binom{r}{\ell}} = \begin{cases} (m_0^{\binom{r}{\ell}} - \epsilon_0^{\binom{r}{\ell}})h_0 + (m_1^{\binom{r}{\ell}} - \epsilon_1^{\binom{r}{\ell}})\zeta_0(c_\pm, d_\pm)h_1 + (m_2^{\binom{r}{\ell}} - \epsilon_2^{\binom{r}{\ell}})\eta_0(c_\pm, d_\pm)h_2, & j = 0, \\ (m_1^{\binom{r}{\ell}} - \epsilon_1^{\binom{r}{\ell}})h_1 + (m_2^{\binom{r}{\ell}} - \epsilon_2^{\binom{r}{\ell}})\zeta_1(c_\pm, d_\pm)h_2 + m_3^{\binom{r}{\ell}}\eta_1(c_\pm, d_\pm)h_3, & j = 1, \\ (m_2^{\binom{r}{\ell}} - \epsilon_2^{\binom{r}{\ell}})h_2 + m_3^{\binom{r}{\ell}}\zeta_2(c_\pm, d_\pm)h_3 + m_4^{\binom{r}{\ell}}\eta_2(c_\pm, d_\pm)h_4, & j = 2, \\ m_j^{\binom{r}{\ell}}h_j + m_{j+1}^{\binom{r}{\ell}}\zeta_j(c_\pm, d_\pm)h_{j+1} + m_{j+2}^{\binom{r}{\ell}}\eta_j(c_\pm, d_\pm)h_{j+2}, & j = 3, \dots, N. \end{cases}$$

3.4. Solution algorithm

The transformation (3.9) turns the nonhomogeneous boundary conditions (3.2) into the homogeneous boundary conditions (3.4). Problem (3.3) can be rewritten in the form

$$\gamma_1\widetilde{u}(x,y) - \Delta\widetilde{u}(x,y) = f(x,y) - (\gamma_1 - \Delta)(u_e^s + u_e^c) \quad \text{in } \Omega, \tag{3.28}$$

subject to the homogeneous boundary conditions (3.4).

The Legendre–Galerkin approximation of (3.28) subject to (3.4) consists of finding $\widetilde{u}_N \in V_N$ such that

$$\gamma_1\Big(\widetilde{u}_N, v\Big) - \Big(\Delta\widetilde{u}_N, v\Big) = \Big(f - (\gamma_1 - \Delta)(u_{e,N}^s + u_e^c), v\Big) \quad \forall v \in V_N, \tag{3.29}$$

where u_e^c and $u_{e,N}^s$ are as defined in (3.14) and (3.20).

Let us denote

$$\begin{aligned} &\widetilde{u}_N(x,y) = \sum_{k=0}^{N}\sum_{j=0}^{N} u_{kj}\,\phi_k(x, a_\pm, b_\pm)\,\phi_j(y, c_\pm, d_\pm), \\ &f_{kj}^* = \big(f - (\gamma_1 - \Delta)(u_{e,N}^s + u_e^c),\ \phi_k(x, a_\pm, b_\pm)\,\phi_j(y, c_\pm, d_\pm)\big), \\ &U = (u_{kj}), \qquad F^* = (f_{kj}^*), \qquad k, j = 0, 1, \dots, N. \end{aligned} \tag{3.30}$$

Taking $v(x,y) = \phi_\ell(x, a_\pm, b_\pm)\,\phi_m(y, c_\pm, d_\pm)$ in (3.29) for $\ell, m = 0, 1, \dots, N$, then we find that (3.29) is equivalent to the following matrix equation:

$$A_{b_\pm}^{a_\pm}\,U\,B_{d_\pm}^{c_\pm} + B_{b_\pm}^{a_\pm}\,U\,A_{d_\pm}^{c_\pm} + \gamma_1(B_{b_\pm}^{a_\pm}\,U\,B_{d_\pm}^{c_\pm}) = F^*, \tag{3.31}$$

where the stiffness and mass matrices $A_{b_\pm}^{a_\pm}$ and $B_{b_\pm}^{a_\pm}$ are as defined in (2.17) and (2.18) respectively, and $A_{d_\pm}^{c_\pm}$, and $B_{d_\pm}^{c_\pm}$ are their counterparts in the spatial direction y.

The direct solution algorithm here developed for the Robin problem in two dimensions relies upon a tensor product process [8,19], which is defined as follows. Let P and R be two matrices of size $n \times n$ and $m \times m$ respectively. Then their tensor-product is a matrix of size $mn \times mn$.

We can also rewrite the Eq. (3.31) in the following form using the Kronecker matrix algebra (see, [25]):

$$L\mathbf{v} \equiv [A_{b\pm}^{a\pm} \otimes B_{d\pm}^{c\pm} + B_{b\pm}^{a\pm} \otimes A_{d\pm}^{c\pm} + \gamma_1(B_{b\pm}^{a\pm} \otimes B_{d\pm}^{c\pm})]\mathbf{v} = \mathbf{f}, \tag{3.32}$$

where $\mathbf{f}$ and $\mathbf{v}$ are F^* and U respectively written in a column vector, i.e.,

$$\mathbf{f} = (f_{00}^*, f_{10}^*, \dots, f_{N,0}^*; f_{01}^*, f_{11}^*, \dots, f_{N,1}^*; \dots; f_{0,N}^*, \dots, f_{N,N}^*)^T, \tag{3.33}$$

$$\mathbf{v} = (u_{00}, u_{10}, \dots, u_{N,0}; u_{01}, u_{11}, \dots, u_{N,1}; \dots; u_{0,N}, \dots, u_{N,N})^T. \tag{3.34}$$

In summary, solving the system (3.1)–(3.2) consists of the following steps:

1. Compute the matrices $A_{b\pm}^{a\pm},\ B_{b\pm}^{a\pm},\ A_{d\pm}^{c\pm},\ B_{d\pm}^{c\pm}$ and F^*.
2. Compute the tensor products $A_{b\pm}^{a\pm} \otimes B_{d\pm}^{c\pm},\ B_{b\pm}^{a\pm} \otimes B_{d\pm}^{c\pm}$ and $B_{b\pm}^{a\pm} \otimes B_{d\pm}^{c\pm}$.
3. Determine the coefficients $\{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4\}$ from (3.15) and then find the corner component $u_e^c(x, y)$.
4. Determine the unknown expansion coefficients $\{d_k^{\mathrm{I}},\ d_k^{\mathrm{II}},\ d_k^{\mathrm{III}}, d_k^{\mathrm{IV}},\ k = 0, 1, \dots, N\}$ from (3.25)–(3.26) and then find the side component $u_{e,N}^s(x, y)$.
5. Write F^* in a column vector $\mathbf{f}$.
6. Obtain a column vector $\mathbf{v}$ by solving (3.32).
7. Obtain the matrix of coefficients U of the solution component $\widetilde{u}_N(x, y)$.

The complete solution to the original nonhomogeneous Robin boundary value problem is finally obtained:

$$u_N(x, y) = \widetilde{u}_N(x, y) + u_e^c(x, y) + u_{e,N}^s(x, y).$$

To end this section, we comment on the computational effort needed for preforming the previous algorithm.

It is well-known that banded systems arise frequently in staged operations and in discretization of differential equations. The advantage is that for banded systems, the fill-in stays within the band and accordingly there is no need for extra storage and much less work is required. Since the matrices $A_{b\pm}^{a\pm},\ B_{b\pm}^{a\pm},\ A_{d\pm}^{c\pm}$ and $B_{d\pm}^{c\pm}$ have sparse structure, then the numerical properties for solving (3.32) by using the LUdecomposition improved.

Now, Step 2 consists of finding the tensor product of the one-dimensional matrices. From the structure of $A_{b\pm}^{a\pm},\ B_{b\pm}^{a\pm},\ A_{d\pm}^{c\pm}$ and $B_{d\pm}^{c\pm}$, hence it takes only $O(N^2)$ operations. Step 3 consists of finding the coefficients $\alpha_1,\ \alpha_2,\ \alpha_3$ and α_4 by solving four linear equations. In Step 4, we solve four pentadiagonal systems each of order $N + 1$ by using the LUdecomposition technique which can be performed in $O(N)$ operations. In Step 6, we solve system (3.32) by using again the LUtechnique which needs $O(N^3)$ operations. Therefore the total computational cost of the algorithm is $O(N^3)$ arithmetic operations.

4. Numerical results and comparisons

We report in this section some numerical results obtained with the algorithms presented in the previous sections.

We consider the following examples.

Example 1. Consider the one-dimensional Neumann problem

$$\frac{3}{2}\, u(x) - u''(x) = f(x), \quad \text{in } I,$$

with exact solution $u(x) = \mathrm{e}^x$.

Similar problems were also investigated by Auteri et al. [16] using a Legendre–Galerkin method (LGM [16]) and Doha et al. [19] using a Jacobi–Galerkin method. Table 1 lists the L^2-and H^1-errors, using the LGM with various choices of N. Numerical results for this problem show that the LGM converges exponentially. We contrast our results with the corresponding results for LGM [16] which we have presented in the third and fifth columns of this table.

Example 2. Consider the nonhomogeneous Dirichlet problem (see the second example in Section 3.4 of [22])

$$\gamma_1 u(x, y) - \Delta u(x, y) = f(x, y), \quad \text{in } \Omega = I \times I, \qquad u|_{\partial\Omega} = e,$$

with exact solution $u(x, y) = x^2 + e^{2x+y}$.

Table 1
One-dimensional problem with Neumann conditions.

N	L^2-error		H^1-error	
	LGM	LGM [16]	LGM	LGM [16]
4	$6.68.10^{-6}$	$2.22\cdot10^{-4}$	$5.39\cdot10^{-5}$	$1.88\cdot10^{-3}$
8	$4.15.10^{-11}$	$7.91\cdot10^{-9}$	$5.85\cdot10^{-10}$	$1.34\cdot10^{-7}$
16	$2.41.10^{-16}$	$3.10\cdot10^{-14}$	$3.16\cdot10^{-16}$	$1.25\cdot10^{-13}$
32	$2.29\cdot10^{-16}$	$9.63\cdot10^{-14}$	$1.66\cdot10^{-16}$	$1.19\cdot10^{-12}$

Table 2
Maximum pointwise error of $u - u_N$ for $N = 8, 16, 32$.

N	LGM	LGM [22]
8	$1.74\cdot10^{-7}$	$2.32\cdot10^{-5}$
16	$5.39\cdot10^{-15}$	$4.44\cdot10^{-13}$
32	$3.99\cdot10^{-15}$	$1.04\cdot10^{-12}$

Table 3
L^∞-error of $u - u_N$ for $N = 8, 12, 16$.

N	Case 1	Case 2	Case 3	LGM [16] (Case 3)	Case 4
8	$4.08.10^{-7}$	$1.88\cdot10^{-7}$	$2.47\cdot10^{-7}$	$(3.1\cdot10^{-5})$	$2.57\cdot10^{-7}$
12	$1.20.10^{-11}$	$6.20\cdot10^{-12}$	$7.21\cdot10^{-12}$	$(4.3\cdot10^{-9})$	$7.10\cdot10^{-12}$
16	$9.83.10^{-15}$	$7.10\cdot10^{-15}$	$7.59\cdot10^{-15}$	$(2.0\cdot10^{-13})$	$1.33\cdot10^{-14}$

This problem is solved in [22] using a Legendre Galerkin method in which the integrals on the right-hand side in the resulting linear system are approximated using $(N + 1)$-point Legendre Gauss quadrature. The maximum pointwise errors of the proposed LGM using a tensor product process are compared in Table 2 with the result provided by LGM [22] using a double diagonalization process which we have presented in the third column of this table. We should note that for all values of N, the present method is always more accurate than the result of LGM [22], which shows the spectral accuracy of our method.

It should be noted that the Dirichlet boundary conditions in this example are nonhomogeneous. In [22], the nonhomogeneous Dirichlet boundary conditions at the corners are dealt with using collocation. Then, the nonhomogeneous Dirichlet boundary conditions on each side, excluding the endpoints, are treated using the L^2 projection onto the space of polynomials vanishing at the endpoints. In [7], the nonhomogeneous Dirichlet boundary conditions are handled by determining two functions u^1 and u^2, defined on $\partial\Omega$, such that $u = u^1 + u^2$ on $\partial\Omega$.

Example 3. Consider the problem

$$u(x, y) - \Delta u(x, y) = f(x, y), \quad \text{in } \Omega = I \times I,$$

with exact solution $u = x^2 + e^{x+2y}$, and the four sets of nonhomogeneous boundary conditions:

- Case 1:

$$\begin{aligned} &u(1, y) = e^r(y), \qquad u(-1, y) = e^\ell(y),\\ &u_y(x, 1) = e^t(x), \qquad u_y(x, -1) = e^b(x). \end{aligned}$$

- Case 2:

$$\begin{aligned} &u_x(1, y) = e^r(y), \qquad u_x(-1, y) = e^\ell(y),\\ &u(x, 1) = e^t(x), \qquad u(x, -1) = e^b(x). \end{aligned}$$

- Case 3:

$$\begin{aligned} &u_x(1, y) = e^r(y), \qquad u_x(-1, y) = e^\ell(y),\\ &u_y(x, 1) = e^t(x), \qquad u_y(x, -1) = e^b(x). \end{aligned}$$

- Case 4:

$$\begin{aligned} &u(1, y) + u_x(1, y) = e^r(y), \qquad 2u(1, y) - u_x(-1, y) = e^\ell(y),\\ &2u_y(x, 1) - u_y(x, 1) = e^t(x), \qquad -u_y(x, -1) + u_y(x, -1) = e^b(x). \end{aligned}$$

In Table 3, we present the maximum pointwise errors of $u - u_N$ using the LGM with various choices of N for the four cases of boundary conditions.

Table 4
Maximum pointwise error of $u - u_N$ for $N = 16, 24, 32$.

N	LGM	Ulta-tau [20]
16	$8.30.10^{-3}$	$7.75.10^{-2}$
24	$4.19.10^{-7}$	$1.06.10^{-5}$
32	$1.17.10^{-11}$	$5.18.10^{-10}$

The Neumann problem (Case 3) is solved in [16] using the Legendre–Galerkin method based on a double diagonalization process. The results provided by LGM [16] are presented in parentheses in the fifth column of Table 3. For these results the integrals defining the perturbed Neumann condition on the four sides were evaluated using the Gauss–Legendre quadrature formula with $N + 1$ points, and the term involving the integral of f was evaluated numerically by means of a suitable Gauss–Legendre quadrature formula. The corresponding integrals in our method are computed numerically using the Gauss–Lobatto Legendre quadrature formula.

Example 4. Consider the Robin problem (see the second example in Section 6 of Doha and Abd-Elhameed [20] (Ulta-tau [20]))

$$-\Delta u = 32\pi^2 \sin(4\pi x) \sin(4\pi y), \quad \text{in } \Omega = I \times I,$$

subject to the boundary conditions

$$u(\pm 1, y) \pm u_x(\pm 1, y) = \pm 4\pi \ \sin(4\pi y),$$
$$u(x, \pm 1) \pm u_y(x, \pm 1) = \pm 4\pi \ \sin(4\pi x).$$

The maximum pointwise errors of the proposed LGM are compared in Table 4 with the result provided by Doha and Abd-Elhameed [20] (Ulta-tau [20]) using a double ultraspherical tau method for its Legendre polynomial case ($\alpha = \frac{1}{2}$).

5. Concluding remarks

We have presented some efficient direct solvers for second-order elliptic problems subject to Robin boundary conditions using the LGM in one and two space variables. In this paper we have found that the matrix elements of the discrete operators are provided explicitly: the mass matrix is pentadiagonal, while the stiffness matrix is diagonal, and this greatly simplifies the steps for obtaining the solutions for these equations.

Although we concentrated on applying our algorithms to solve constant coefficient differential equations, we do claim that such algorithms can be applied to solve differential equations with polynomial coefficients of any order. Our algorithms for the second-order equations are very efficient and numerically stable. Numerical results are presented which exhibit the high accuracy of the proposed algorithms.

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